Asymptotic density of collision orbits in the Restricted Planar Circular 3 Body Problem

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The 3 body problem

- Consider three bodies $q_1$, $q_2$ and $q_3$ with masses $m_1, m_2, m_3 > 0$, 

\[
\frac{d^2 q_i}{dt^2} = \sum_{j=1, j\neq i}^3 m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \quad q_i \in \mathbb{R}^3
\]

- Long term behavior?

- Chazy (1922): Final motions – Behavior of the bodies $q_k(t)$ as $t \to \pm \infty$. 
Chazy classification

- Types of final motions:
  - $\mathcal{H}^+: |r_k| \to \infty, |\dot{r}_k| \to c_k \neq 0$ as $t \to +\infty$;
  - $\mathcal{H}P_k^+: |r_k| \to \infty, |\dot{r}_k| \to 0, |\dot{r}_i| \to c_i > 0$ ($i \neq k$);
  - $\mathcal{H}E_k^+: |r_k| \to \infty, |\dot{r}_i| \to c_i > 0$ ($i \neq k$), $\sup_{t \geq 0} |r_k| < \infty$;
  - $\mathcal{P}E_k^+: |r_k| \to \infty, |\dot{r}_i| \to 0$ ($i \neq k$), $\sup_{t \geq 0} |r_k| < \infty$;
  - $\mathcal{P}_+: |r_k| \to \infty, |\dot{r}_k| \to 0$;
  - $\mathcal{B}^+: \sup_{t \geq 0} |r_k| < \infty$;
  - $\mathcal{OS}^+: \limsup_{t \to \infty} \max_k |r_k| = \infty, \liminf_{t \to \infty} \max_k |r_k| < \infty$.

- Classification for trajectories defined for all time.

- Some orbits are not defined for all time: orbits hitting collisions.
Collision orbits

\[
\frac{d^2 q_i}{dt^2} = \sum_{j=1, j \neq i}^{3} m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \quad q_i \in \mathbb{R}^2
\]

- **Collision set:** \( C = \{q_1 = q_2\} \cup \{q_1 = q_3\} \cup \{q_2 = q_3\} \)

- **Collision orbit:** orbit which hits a collision at some time \( t = t^* \).
Herman conjecture

- Fix the center of mass at the origin.
- Reparameterize the flow so that it takes infinite time to get to collision.
- **Non-wandering set:** Consider a dynamical system $\phi : X \to X$, $x \in X$ is non-wandering if for every open neighborhood $U$ of $x$ and any $N$ satisfies $\phi^n(U) \cap U \neq \emptyset$ for some $n > N$.
- **Herman question:** Is the non-wandering set nowhere dense in all energy levels?
- In particular: Is the set of bounded orbits nowhere dense?
How abundant/rare are collision orbits?

- **Saari 1970’s (also Fleischer & Knauf 2018):** The set of collision orbits has measure zero.

- **Alexeev conjecture (1981):** Is there an open set $\mathcal{U}$ in phase space possessing a dense subset $\mathcal{D} \subset \mathcal{U}$ whose points lead to collision?

  This conjecture goes back to Siegel.

- If Alexeev conjecture is true, would imply a dense set of bounded orbits.

- Could Alexeev conjecture lead to a negative answer to Herman conjecture?

- To understand Alexeev conjecture: consider the case $m_2 = m_3 = 0$. 
The case: $m_2 = m_3 = 0$

- Body 1 does not move.
- Body 2 and 3

$$\frac{d^2 q_i}{dt^2} = m_1 \frac{q_1 - q_i}{\|q_1 - q_i\|^3}$$

form two uncoupled 2BPs.
- Place them on ellipses.

- Take ellipses that intersect transversally.
- They form an open set in phase space foliated by 2-tori.
- All solutions are either periodic or quasi-periodic.
The case: $m_2 = m_3 = 0$

- If periods of $q_2$ and $q_3$ are inconmensurable, collision orbits are dense in this $\mathbb{T}^2$.
- Periods is $2\pi a^{3/2}$ where $a$ is the semimajor axis of the ellipse.
- For a dense set of $a$’s the periods are inconmensurable.
- Tori with dense collision orbits are dense in an open set.
General case: $m_2, m_3 > 0$

- Alexeev: Does density still hold?
- For $m_2, m_3 > 0$ small, this is not a regular perturbation problem.
- The system blows up in a small neighborhood of collisions.
- We consider a simpler model: The Restricted Planar Circular 3 Body problem.
The Restricted Planar Circular 3 Body problem

- Three bodies of masses $1 - \mu$, $\mu$ and 0 under the effects of the Newtonian gravitational force.
- Primaries $q_1$ and $q_2$ orbiting on circles.
- Rotating coordinates:
  - Primaries at $q = (-\mu, 0)$ and $q = (1 - \mu, 0)$
  - Dynamics of the third body $q$ is given by the 2 dof Hamiltonian

$$H(q, p, t) = \frac{\|p\|^2}{2} - (p_2 q_1 - p_1 q_2) - \frac{1 - \mu}{\|q + \mu\|} - \frac{\mu}{\|q - (1 - \mu)\|}$$

- Phase space: $\mathbb{R}^4 \setminus \{\text{collisions}\}$. 
Collisions are asymptotically dense

Theorem (M. G. – V. Kaloshin – J. Zhang)

Consider the RPC3BP. There exists an open set $\mathcal{U} \subset \mathbb{R}^4$ and $\tau > 0$, independent of $\mu$, such that, for $\mu$ small enough, there is a $\mu^\tau$-dense set $\mathcal{D} \subset \mathcal{U}$ whose points lead to collision.

- $\mu^\tau$ dense $\equiv \mu^\tau$ neighborhoods of all points in $\mathcal{D}$ cover $\mathcal{U}$.
- So far $\tau$ can be taken $\tau = \frac{1}{17 + \sigma}$ for any $\sigma > 0$.
- $\mathcal{U}$ gives open sets in the energy level $H = h$ for energies

$$h \in \left(-\frac{3}{2}, \sqrt{2}\right).$$
The set $\mathcal{U}$

- The set $\mathcal{U}$ can be easily characterized in terms of Delaunay coordinates:
  - $L$ square root of the semimajor axis of the ellipse.
  - $G$ is the angular momentum.
  - $\ell$ is the mean anomaly.
  - $g$ is the argument of the perihelion with respect the primaries line.

- Then $\mathcal{U}$ is the interior of any compact set contained in

$$
\mathcal{V} = \left\{ (\ell, g, L, G) \in \mathbb{T}^2 \times (0, +\infty) \times (-L, 0) \cup (0, L) : \right.
$$

$$
\frac{G^2}{1 + e} < 1 < \frac{G^2}{1 - e}, \quad H(\ell, g, L, G) \in \left( -\frac{3}{2}, \sqrt{2} \right)
$$

where $e = \sqrt{1 - \frac{G^2}{L^2}}$. 

The set $\mathcal{U}$

- $\mathcal{U}$ corresponds to where in the unperturbed case ($\mu = 0$) the ellipses of the two bodies intersect transversally.

- In particular we only consider collisions with the small primary at $(1 - \mu, 0)$.

- The same set were the existence of second species periodic solutions are looked for (Niederman, Marco, Bolotin, McKay,...)

- Collisions with the massive primary – Punctured tori: Chenciner, Llibre, Féjoz, Zhao.
Other almost density results: punctured tori

- Punctured tori also give an almost density result of collision orbits for the RP3BP and the full 3BP.

- However, for the RP3BP the corresponding set $\mathcal{U}$ is very small (either its measure goes to zero as $\mu \to 0$ or two bodies are arbitrarily close).

- For the full 3BP almost density in “big” sets if one places one of the bodies very far away.
A disproof of a weak version of Herman conjecture

- **Herman question**: Is the non-wandering set nowhere dense in all energy levels?

- Consider a dynamical system $\{\phi_t\}_{t \in \mathbb{R}}$ defined on a topological space $X$. Then, a point $x \in X$ is called $\delta$-non-wandering, if for any neighborhood $V$ of it containing the $\delta$-ball $B_\delta(x)$, there exists $T > 1$ such that $\phi_T(V) \cap V \neq \emptyset$.

**Theorem (M. G. – V. Kaloshin – J. Zhang)**

Any point belonging to the open set $U$ is $O(\mu^T)$–non wandering under the flow $\phi_t$ of the RPC3BP. More concretely, for any $z \in U$, we can find a $O(\mu^T)$-neighborhood $V_\mu$ of it and times $0 < T'_\mu < T_\mu$ such that $\phi_{T'_\mu}(V_\mu)$ is $O(\mu^T)$–close to a collision and $\phi_{T_\mu}(V_\mu) \cap V_\mu \neq \emptyset$. 
Summarizing

For the RPC3BP, at $\mu^\tau$ scales for $\tau \leq \frac{1}{17 + \sigma}$, $\sigma > 0$:

- Alexeev conjecture is correct
- Herman conjecture is not.

What happens for the true conjectures for the full 3BP?

Are they incompatible?
Some ideas of the proof of almost density of collisions

- Take any point \( P \in \mathcal{U} \): we want to find \( Q \) \( \mu^\tau \)-close to it hitting a collision.

- **Case \( \mu = 0 \):**
  - \( \mathcal{U} \) foliated by 2 dimensional tori.
  - Choose \( Q \) in an orbit in a non-resonant torus hitting collision (they are dense).

- \( Q \) may need a very long time to hit collision.

- **Case \( \mu > 0 \):** Choose a \( \mu^{3\tau} \)-long curve \( \mu^\tau \)-close to \( P \) and show that a point in this curve hits a collision.
Some ideas of the proof: three regimes

1. **Far from collision (points $\mu^{3\tau}$ away from collision)** the zero mass body $q$ (basically) only notices the main primary: nearly integrable setting.

2. **Transition zone**: $q$ notices the two primaries but orbits spend there very short time.

3. **Small neighborhood of the collision ($\rho \mu^{1/2}$ neighborhood of the collision with $\rho \gg 1$)**: $q$ (basically) only notices the small primary – A different nearly integrable setting.
Regime 1: far from collision

- We are in a nearly integrable regime.
- Problem: the point may need a very long time to reach Regime 2.
- We apply KAM.
- KAM is global: it cannot be applied directly due to the collisions (the Hamiltonian blows up at collision!)
- Remove the collision by multiplying $H$ by a bump function supported at $\mu^{3\tau}$-ball centered at the collision.
- The modified Hamiltonian is close to a 2 body problem (in low regularity).
Regime 1: far from collision

- We want to apply KAM with lowest possible regularity: the more regularity, the worse estimate on the Hamiltonian with bump functions.

- Constant type frequencies are $\gamma$-dense

\[ |q\omega - p| \geq \frac{\gamma}{|q|}. \]

- We apply Herman version of KAM (for $C^{3+\sigma}$ maps and constant type frequencies): tori are $\gamma$-dense.

- Each torus has two (removed) collisions.

- Orbits on the tori are true orbits of the RPC3BP as long as do not intersect a $\mu^{3\tau}$ neighborhood of the collisions.
Regime 1: How to reach well Regime 2

We wanted: any point $P$ has a $\mu^{3\tau}$-long curve $\mu^{\tau}$-close to it and a point in this curve hits collision.

Take a KAM torus $\mu^{\tau}$ close to $P$ and $\mu^{3\tau}$-long curve in this torus
Regime 1: How to reach well Regime 2

- The forward orbit of the small curve has to hits “well” the puncture around one of the collision so that it can be sent forward to Regimes 2 and 3.

- **Well:**
  - The image of the segment hits the half of the boundary of the neighborhood where the velocity is pointing inwards.
  - The orbit cannot have intersected before the punctures around collisions (we want a true orbit of RPC3BP!).

- Moreover: the tangent vectors at the hitting points are close to parallel and velocity is of order $\sim 1$. 
Regime 1: far from collision

- We want to optimize the density coefficient
- Small $\gamma$: gives better density of tori.
- To have the segment hitting well we need to avoid close encounters with collisions before a good hitting.
- We need a strong Diophantine condition $\rightarrow \gamma$ big.
- KAM + Non-homogeneous Dirichlet Theorem leads to

$$\gamma = \mu^\tau \quad \text{with} \quad \tau = \frac{1}{17 + \sigma}, \quad \sigma > 0.$$
Regime 2

- Regime 2: $\mu^{3\tau}$-close to collision and $\rho\mu^{1/2}$-far to collision with $\rho \gg 1$.

- It is a small annulus of width $\mu^{3\tau}$ where the two bodies are “not too close”.

- We use the true RPC3BP.

- Velocity of order $\sim 1$ (collisions are “far enough” to control it).

- Thus: the flow is almost tubular.

- Conclusion: the propagated segment goes from the outer to the inner boundary with almost constant velocity.
Regime 3

- The influence of the small primary is dominant.

- Flow far from tubular and close to a new 2 body problem (close to collision).

- Apply Levi-Civita regularization

- Analyze backward orbits departing from collisions
In (scaled) Levi-Civita coordinates, the RPC3BP becomes

\[ K(z, w) = \frac{1}{2}(|w|^2 - |z|^2) + \mu^{1/2}O_4(z, w) \]

where \( z = 0 \) is the collision set.

- Run backwards the collision orbits to the boundary between Regimes 2 and 3.
- Restricting to the level of energy, they give a curve at the boundary.
- Consider the incoming curve from Regime 2 in these coordinates.
- Plot these two curves in the plane \((\text{arg}(z), \text{arg}(w))\).
The collision orbit

- **Blue**: the incoming curve coming from Regime 1 and 2.
- **Red**: backward orbits of collision orbits.

They are both $C^0$ curves: they must intersect.